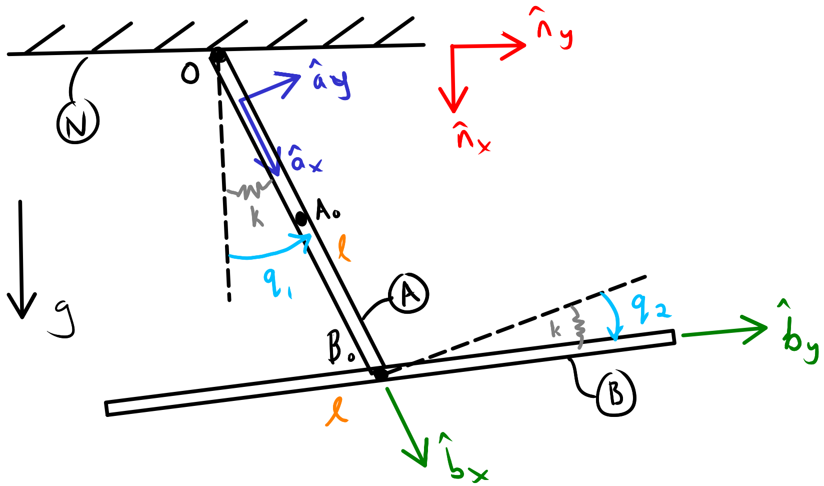


O
N
$\hat{a}_y$
$\hat{a}_x$
$\hat{n}_y$
$\hat{n}_x$
k
A_o
ℓ
q_1
A
g
B_o
k
q_2
$\hat{b}_y$
B
ℓ
$\hat{b}_x$

Generalized Forces

Partial Velocities

Every velocity and angular velocity expressed in terms of generalized speeds can be written like so;

holonomic system with G.Ss $u_1, \dots, u_n$ in a reference frame N

r^{th} generalized

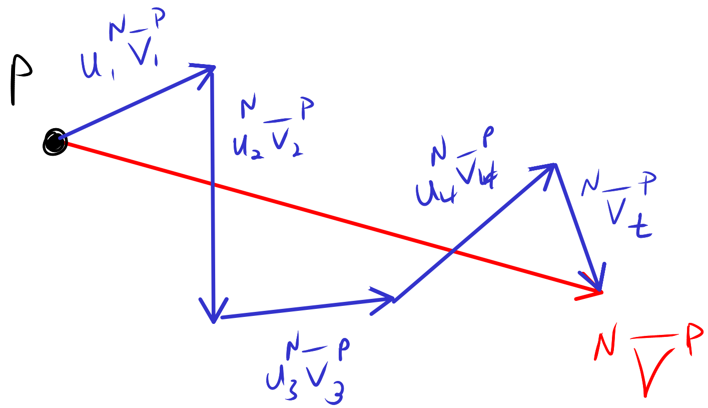
$$
{}^N\bar{V}^P = \sum_{r=1}^{n} u_r \, {}^N\bar{V}_r^P + {}^N\bar{V}_t^P
$$

remainder

r^{th} partial velocity of P in N

$$
{}^N\bar{\omega}^B = \sum_{r=1}^{n} u_r \, {}^N\bar{\omega}_r^B + {}^N\bar{\omega}_t^P
$$

r^{th} partial angular velocity in N

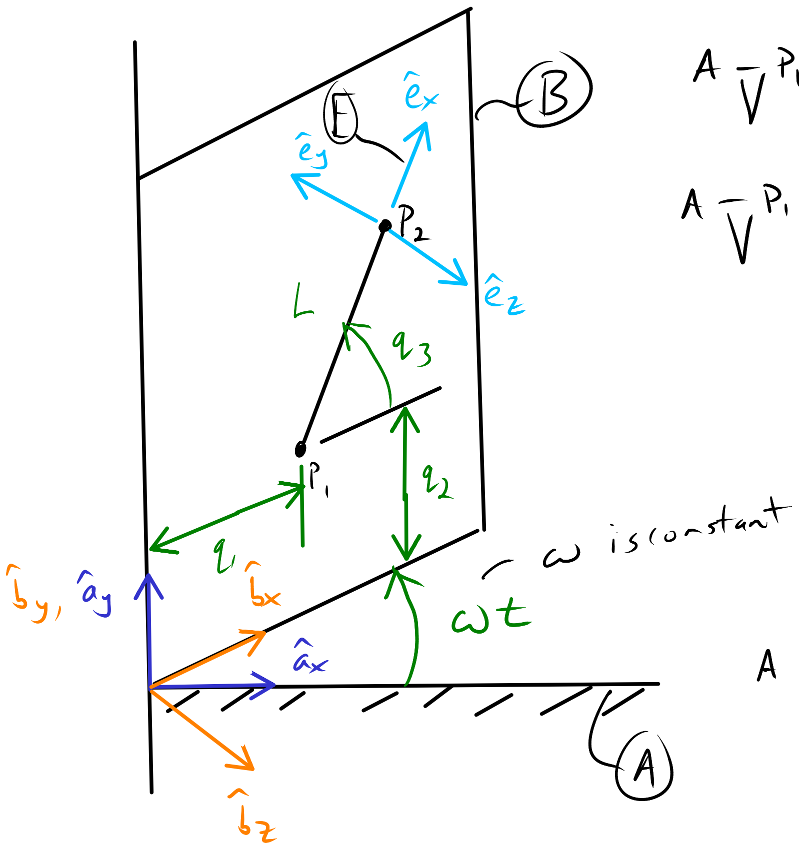


${}^N\bar{V}_r^P$ show how ${}^N\bar{V}^P$ will change if u_r changes.

${}^N\bar{V}_1^P, \ldots, {}^N\bar{V}_n^P$ are the sensitivities of ${}^N\bar{V}^P$ to unit changes in $u_1, \ldots, u_n$.

$${}^N\bar{V}_r^P = \frac{{}^N\partial\, {}^N\bar{V}^P}{\partial u_r} \qquad {}^N\bar{\omega}_r^A = \frac{{}^N\partial\, {}^N\bar{\omega}^B}{\partial u_r}$$

Example Fig 2.6.1 in Kane & Levinson's book pg 29



$$^A\bar{V}^{P_1} = \dot{q}_1 \hat{b}_x + \dot{q}_2 \hat{b}_y - \omega q_1 \hat{b}_z$$

$$^A\bar{V}^{P_1} = (\dot{q}_1 c_3 + \dot{q}_2 s_3)\hat{e}_x + (-\dot{q}_1 s_3 + \dot{q}_2 c_3)\hat{e}_y - \omega q_1 \hat{e}_z$$

$$^A\bar{V}^{P_1} = (\dot{q}_1 \cos\omega t - \omega q_1 \sin\omega t)\hat{a}_x + \dot{q}_2 \hat{a}_y - (\dot{q}_1 \sin\omega t + \omega q_1 \cos\omega t)\hat{a}_z$$

Case 1 $u_1 = \dot{q}_1$, $u_2 = \dot{q}_2$, $u_3 = \dot{q}_3$

$$^A\bar{V}^{P_1} = u_1 \hat{b}_x + u_2 \hat{b}_y - \omega q_1 \hat{b}_z$$

$$^A\bar{V}_1^{P_1} = \hat{b}_x, \quad ^A\bar{V}_2^{P_1} = \hat{b}_y, \quad ^A\bar{V}_3^{P_1} = 0, \quad ^A\bar{V}_t^{P_1} = -\omega q_1 \hat{b}_z$$

Case 2 $u_1 = \dot{q}_1 c_3 + \dot{q}_2 s_3$, $u_2 = \dot{q}_2 c_3 - \dot{q}_1 s_3$, $u_3 = \dot{q}_3$

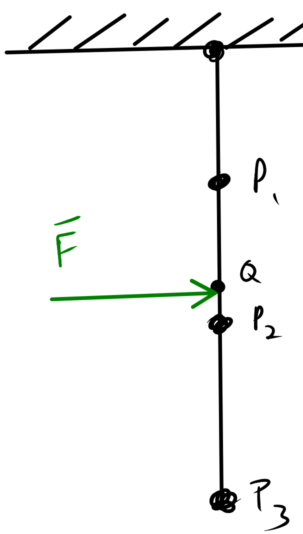
$$^A\bar{V}^{P_1} = u_1 \hat{e}_x + u_2 \hat{e}_y - \omega q_1 \hat{e}_z$$

$$^A\bar{V}_1^{P_1} = \hat{e}_x, \quad ^A\bar{V}_2^{P_1} = \hat{e}_y, \quad ^A\bar{V}_3^{P_1} = 0, \quad ^A\bar{V}_t^{P_1} = -\omega q_1 \hat{e}_z$$

$$ {}^{A}\bar{\omega}^{E} = {}^{A}\bar{\omega}^{B} + {}^{B}\bar{\omega}^{E} = \omega s_3 \hat{e}_x + \omega c_3 \hat{e}_y + u_3 \hat{e}_z $$

$$ {}^{A}\bar{\omega}_1^{E} = 0, \quad {}^{A}\bar{\omega}_2^{E} = 0, \quad {}^{A}\bar{\omega}_3^{E} = \hat{e}_z, \quad {}^{A}\bar{\omega}_t^{E} = \omega s_3 \hat{e}_x + \omega c_3 \hat{e}_y $$

Given a holonomic multibody system, a force applied to the system, in general, will cause everything to move.



angles: q_i

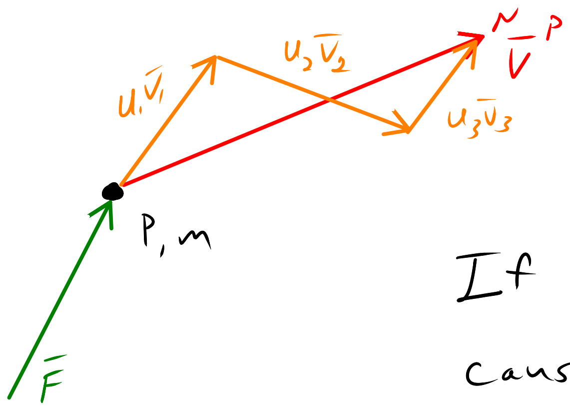
$u_i = \dot{q}_i$

If we apply $\bar{F}$ @ Q then all of the u's will change. But how much will they change? Partial velocities are the answer to this question.

Suppose ${}^N\bar{V}^P$ is the velocity of a particle of mass m acted on by a force $\bar{F}$ that causes an acceleration ${}^N\bar{a}^P = \frac{{}^N d\,{}^N\bar{V}^P}{dt}$ which occurs with changes in the generalized speeds (u's).

Because ${}^N\bar{V}^P = \sum u_r \bar{V}_r + \bar{V}_t$

${}^N\bar{V}_r^P$ is the direction (or component) of $\bar{F}$ that causes u_r to change ($\dot{u}_r$).



$$\frac{\bar{F}}{m} \cdot {}^N\bar{V}_r^P \Rightarrow \dot{u}_r = f(\bar{u}, \bar{q}, t)$$

If $\bar{V}_r \perp \bar{F}$, then $\bar{F}$ doesn't cause u_r to change.

Generalized Active Forces

Suppose we have a holonomic system of ν particles with generalized speeds $u_1, \ldots, u_n$ and n degrees of freedom in reference frame A. Then we define the r^{th} holonomic generalized active force in A as:

$$F_r \triangleq \sum_{i=1}^{\nu} {}^A\bar{V}_r^{P_i} \cdot \bar{R}_i \quad \text{for } r = 1, \ldots, n$$

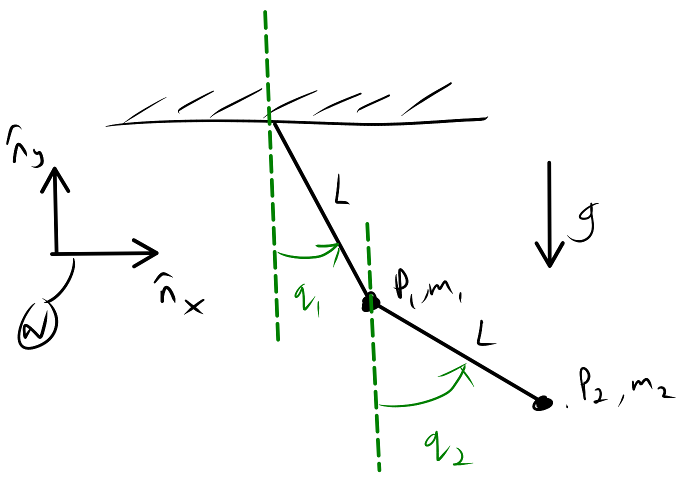
r^{th} holonomic partial velocity

resultant force on P_i

1. r^{th} GAF is scalar value
2. contains contributions from all particles unless ${}^A\bar{V}_r^{P_i} \perp \bar{R}_i$
3. F_r corresponds to u_r, e.g. $F_3 \rightarrow u_3$

F_r captures the forcing terms to the equations of motion, i.e. how the contributing forces cause change in generalized speeds.

Example: double simple pendulum



use $u_1 = \dot{q}_1$ and $u_2 = \dot{q}_2$

Find F_1 and F_2

Velocities

$$^N\bar{V}^{P_1} = L\dot{q}_1(c_1\hat{n}_x + s_1\hat{n}_y)$$

$$^N\bar{V}^{P_2} = {}^N\bar{V}^{P_1} + L\dot{q}_2(c_2\hat{n}_x + s_2\hat{n}_y)$$

Partial velocities

$$^N\bar{V}_1^{P_1} = L(c_1\hat{n}_x + s_1\hat{n}_y)$$

$$^N\bar{V}_2^{P_1} = 0$$

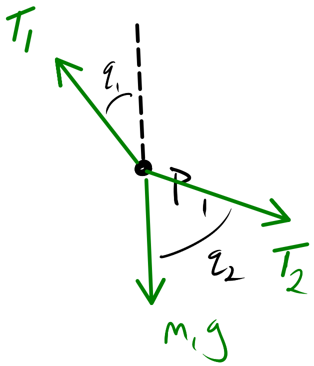
$$^N\bar{V}_1^{P_2} = L(c_1\hat{n}_x + s_1\hat{n}_y)$$

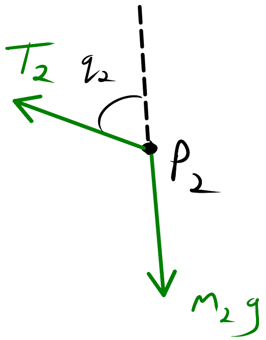
$$^N\bar{V}_2^{P_2} = L(c_2\hat{n}_x + s_2\hat{n}_y)$$

Forces acting on the two particles.

Draw FBDs

T_1 & T_2 are <u>distance</u> <u>forces</u>.





$$\bar{R}_1 = T_1(c_1\hat{n}_y - s_1\hat{n}_x) - m_1 g\hat{n}_y + T_2(-c_2\hat{n}_y + s_2\hat{n}_x)$$

$$\bar{R}_2 = -m_2 g\hat{n}_y - T_2(-c_2\hat{n}_y + s_2\hat{n}_x)$$

$$^N\bar{V}_1^{P_1} = L(c_1\hat{n}_x + s_1\hat{n}_y)$$

$$^N\bar{V}_2^{P_1} = 0$$

$$^N\bar{V}_1^{P_2} = L(c_1\hat{n}_x + s_1\hat{n}_y)$$

$$^N\bar{V}_2^{P_2} = L(c_2\hat{n}_x + s_2\hat{n}_y)$$

Now

Generalized active forces

$$F_1 = {}^N\bar{V}_1^{P_1} \cdot \bar{R}_1 + {}^N\bar{V}_1^{P_2} \cdot \bar{R}_2$$

$$F_1 = Lc_1(\cancel{-T_1 s_1} + \cancel{T_2 s_2}) + Ls_1(\cancel{T_1 c_1} - m_1 g - \cancel{T_2 c_2}) + Lc_1(\cancel{-T_2 s_2}) + Ls_1(-m_2 g + \cancel{T_2 c_2})$$

$$F_1 = -Ls_1 m_1 g - Ls_1 m_2 g = \boxed{-Ls_1 g(m_1 + m_2)}$$

T_1 & T_2 are not present! Not a coincidence.

T_1 & T_2 do not contribute to the GAF.

$$F_2 = \cancel{{}^N\bar{V}_2^{P_1}}^{\,0} \cdot \bar{R}_1 + {}^N\bar{V}_2^{P_2} \cdot \bar{R}_2 = Lc_2(\cancel{-T_2 s_2}) + Ls_2(-m_2 g + \cancel{T_2 c_2})$$

$$\boxed{F_2 = -L s_2 m_2 g}$$

Note the units of GAFs are force × length. This is because u_1, u_2 are angular rates. If u_r is linear speed the F_r will have units force.

The projection of forces onto partial velocities is central to Kane's method.

Generalized Active Force for a Rigid Body

B is a rigid body in a holonomic system with n DoF in A. All forces on B can be represented by a resultant force bound to point Q in B and a torque of a couple.

$$(F_r)_B \triangleq {}^A\bar{V}_r^Q \cdot \bar{R} + {}^A\bar{\omega}_r^B \cdot \bar{T}$$

torque on B

resultant on B bound to a line of action through Q

for single body B

Q can be the mass center of B, but doesn't have to

Generalized Inertia Forces

Given a holonomic system of ν particles with n DoF in A and generalized speeds $u_1, \ldots, u_n$ the rth holonomic generalized inertia force is:

$$F_r^* \triangleq \sum_{i=1}^{\nu} {}^A\bar{V}_r^{P_i} \cdot \bar{R}_i^* \quad \leftarrow \text{inertia force} \qquad \text{where } \bar{R}_i^* \triangleq -m_i\, {}^A\bar{a}^{P_i}$$

If the system includes a rigid body then its contribution to F_r^* is:

mass center of B — inertia force on B — inertia torque on B

$$(F_r^*)_B = {}^A\bar{V}_r^{B_o} \cdot \bar{R}^* + {}^A\bar{\omega}_r^B \cdot \bar{T}^*$$

$$\bar{R}^* \triangleq -m_B\, {}^A\bar{a}^{B_o} \qquad \bar{T}^* \triangleq -\sum_{i=1}^{\beta} m_i\, \bar{r}^{P_i/B_o} \times {}^A\bar{a}^{P_i}$$

P_i for $i=1,\ldots,\beta$ particles that make up body B

m_i is mass of P_i

$\bar{r}^{P_i/B_o}$ $\quad B_o \rightarrow P_i$

${}^A\bar{a}^{P_i}$

For rigid body

$$\bar{T}^* = -{}^A\bar{\alpha}^B \cdot \breve{I}^{B/B_o} - {}^A\bar{\omega}^B \times \breve{I}^{B/B_o} \cdot {}^A\bar{\omega}^B$$

Summary

Generalized active forces map contributing forces to generalized speeds.
↳ "forcing terms"

Generalized inertia forces map inertial effects to generalize speeds.

Newton-Euler Equations for a single rigid body

$$\bar{F} = \frac{d\bar{p}}{dt} \quad \text{Newton's 2nd Law}$$

GAF

$$\bar{M} = \frac{d\bar{H}}{dt} \quad \text{Euler's 2nd Law}$$

GIF