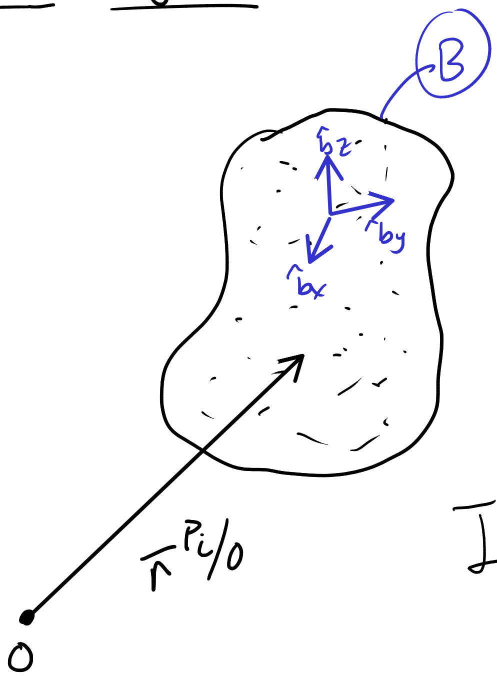
Inertia Dyadic



$$\bar{I}_x^{B/O}, \quad \bar{I}_y^{B/O}, \quad \bar{I}_z^{B/O}$$

3x3

$$\bar{\bar{I}}^{B/O} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}_B$$

symmetric

inertia matrix

inertia tensor

2nd order tensor

$\underline{I_{xy}} = I_{yx}$

see notes

6 unique inertia scalars for body or set of particles B wrt O.

first order tensors

$$\bar{V} = \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix}_B$$

$$\bar{V} = V_x \hat{b}_x + V_y \hat{b}_y + V_z \hat{b}_z$$

Dyadic lets us write 2^{nd} order tensors in this same way as the vectors.

$$
\begin{aligned}
\breve{I}^{B/O} = \; & \overset{\text{scalars}}{I_{xx}} \overbrace{\hat{b}_x \hat{b}_x}^{\text{dyadic}} + I_{xy} \hat{b}_x \hat{b}_y + I_{xz} \hat{b}_x \hat{b}_z + \\
& I_{yx} \hat{b}_y \hat{b}_x + I_{yy} \hat{b}_y \hat{b}_y + I_{yz} \hat{b}_y \hat{b}_z + \\
& I_{zx} \hat{b}_z \hat{b}_x + I_{zy} \hat{b}_z \hat{b}_y + I_{zz} \hat{b}_z \hat{b}_z
\end{aligned}
$$

$\hat{b}_x \hat{b}_x$ is the outer product of two vectors $\hat{b}_x \otimes \hat{b}_x$

$$
\hat{b}_x \hat{b}_x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_B \otimes \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

unit dyad

analagous to unit vector for vectors

$\hat{b}_x \hat{a}_x \Rightarrow$ unit dyad

Unit dyadic

$$
\breve{U} \triangleq \hat{b}_x \hat{b}_x + \hat{b}_y \hat{b}_y + \hat{b}_z \hat{b}_z \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_B
$$

represents identity matrix

With vectors we can write them in terms of unit vectors from different reference frames.

$$\overline{W} = a\hat{b}_x + c\hat{d}_y + d\hat{f}_z$$

Also true for dyadics:

$$\breve{Q} = a\hat{b}_x\hat{b}_x + c\hat{b}_x\hat{a}_y + d\hat{c}_z\hat{c}_z + e\hat{f}_z\hat{a}_x$$

See notes for dyadic identities.

$$\overline{V}\cdot\breve{Q} \neq \breve{Q}\cdot\overline{V}$$ not commutative

$$\overline{V}\cdot\breve{U} = \breve{U}\cdot\overline{V} = \overline{V}$$

unit dyadic

Inertia Dyadic

$\overline{I}_a^{B/o}$ holds all moments & products of inertia, but it's dependent on $\hat{n}_a$

$$\overline{I}_a^{B/o} = \sum_{i=1}^{\nu} m_i \, \overline{r}^{P_i/o} \times (\hat{n}_a \times \overline{r}^{P_i/o})$$

Vector triple product identity: $\overline{a} \times (\overline{b} \times \overline{c}) = \overline{b}(\overline{a} \cdot \overline{c}) - \overline{c}(\overline{a} \cdot \overline{b})$

$$\overline{I}_a^{B/o} = \sum_{i=1}^{\nu} m_i \left[\hat{n}_a (\overline{r}^{P_i/o} \cdot \overline{r}^{P_i/o}) - \overline{r}^{P_i/o} (\overline{r}^{P_i/o} \cdot \hat{n}_a) \right]$$

$$\overline{I}_a^{B/o} = \sum_{i=1}^{\nu} m_i \left[\underbrace{\hat{n}_a \left|\overline{r}^{P_i/o}\right|^2}_{\text{vector}} \cdot \underset{\substack{\uparrow \\ \text{unit} \\ \text{dyadic}}}{\breve{U}} - \underbrace{\left(\overline{r}^{P_i/o} \overline{r}^{P_i/o}\right)}_{\text{dyadic}} \cdot \hat{n}_a \right]$$

$$\overline{I}_a^{B/o} = \underbrace{\left(\sum_{i=1}^{\nu} m_i \left[\left|\overline{r}^{P_i/o}\right|^2 \breve{U} - \overline{r}^{P_i/o} \overline{r}^{P_i/o} \right] \right)}_{\text{inertia dyadic}} \cdot \hat{n}_a$$

$$\bar{I}_a^{B/O} = \underbrace{\breve{I}^{B/O}}_{\text{inertia dyadic}} \cdot \hat{n}_a$$

$\breve{I}^{B/O}$ contains all moments & products of inertia of B wrt O in a <u>basis</u> <u>independent</u> way. ($\hat{n}_a$ is not needed)

$$\breve{I}^{B/O} = I_{xx}\,\hat{b}_x\hat{b}_x + I_{xy}\,\hat{b}_x\hat{b}_y + \ldots + I_{zx}\,\hat{b}_z\hat{b}_x$$

If O is B_o (mass center of B)

$\breve{I}^{B/B_o}$ "central inertia dyadic"

Working with dyadics

Expressing an inertia dyadic in any reference frame.

$$[\mathbf{I}]_B = {}^B C^A \, [\mathbf{I}]_A \, {}^A C^B \qquad [\bar{v}]_B = {}^B C^A \, [\bar{v}]_A$$

.express( )

Angular Momentum

$${}^{A}\overline{H}^{B/O} \triangleq \underset{\smile}{I}^{B/O} \cdot {}^{A}\overline{\omega}^{B}$$

angular momentum of B about O in A

$${}^{A}\overline{H}^{B/B_O} = \underset{\smile}{I}^{B/B_O} \cdot {}^{A}\overline{\omega}^{B}$$

"central angular momentum"

Principal Axes

If products of inertia are zero for body B about any mutaully perpendicular unit vectors.

$$\underset{\smile}{I}^{B/B_O} = I_{11}\hat{b}_1\hat{b}_1 + I_{22}\hat{b}_2\hat{b}_2 + I_{33}\hat{b}_3\hat{b}_3$$

inertially symmetric object wrt $\hat{b}_1, \hat{b}_2, \hat{b}_3$

$\hat{b}_1, \hat{b}_2, \hat{b}_3$ principal axes of B

I_{11}, I_{22}, I_{33} principal moments of inertia of B

$$\sum \bar{M} = \frac{d\bar{H}}{dt}$$

Porrallel Axis Theorem

$$\breve{\underline{I}}^{B/O} = \breve{\underline{I}}^{B/B_O} + \breve{\underline{I}}^{B_O/O}$$

inertia dyadic of B about O

central inertia dyadic of B

inertia dyadic of a particle of mass m (mass of B) @ B_O with respect point O

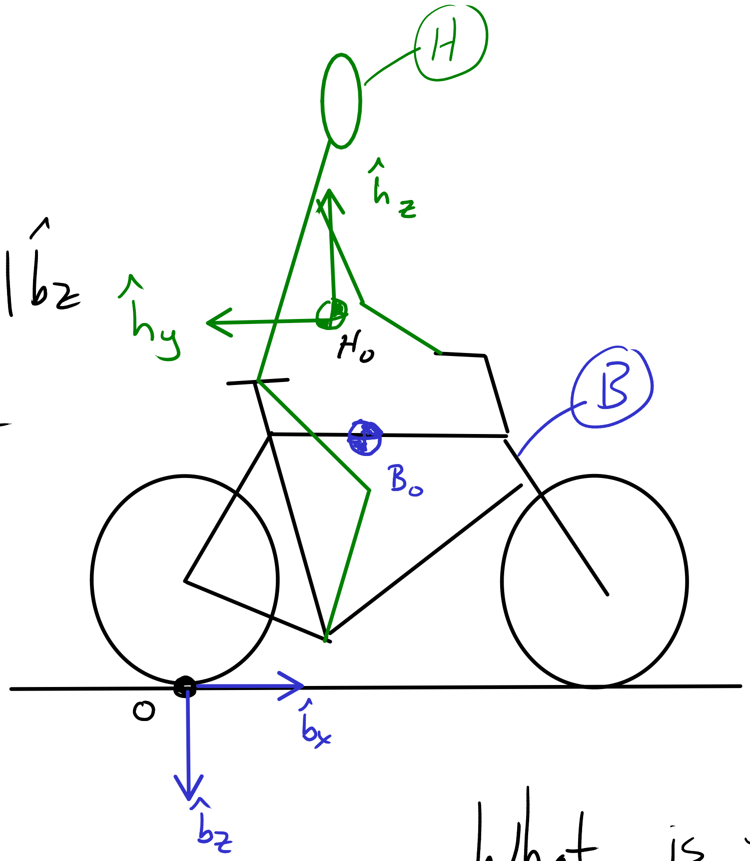
$$\breve{\underline{I}}^{B_O/O} = m\left(\left|\bar{r}^{B_O/O}\right|^2 \breve{U} - \bar{r}^{B_O/O}\,\bar{r}^{B_O/O}\right)$$

Example

$m_H = 84$ kg

$\bar{r}^{H_o/O} = 0.4\,\hat{b}_x - 1.1\,\hat{b}_z$

$$\bar{I}^{H/H_o} = 11.3\,\hat{h}_x\hat{h}_x + 11.0\,\hat{h}_y\hat{h}_y + 2.3\,\hat{h}_z\hat{h}_z - 1.7\,\hat{h}_y\hat{h}_z - 1.7\,\hat{h}_z\hat{h}_y$$



$m_B = 9.9$ kg

$\bar{r}^{B_o/O} = 0.3\,\hat{b}_x - 0.5\,\hat{b}_z$ m

$$\bar{I}^{B/B_o} = 0.5\,\hat{b}_x\hat{b}_x + 1.3\,\hat{b}_y\hat{b}_y + 0.8\,\hat{b}_z\hat{b}_z - 0.1\,\hat{b}_x\hat{b}_z - 0.1\,\hat{b}_z\hat{b}_x \quad \text{kg m}^2$$

$${}^{B}C^{H} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

What is the inertia of $\underbrace{B+H}_{F}$ about F_o?