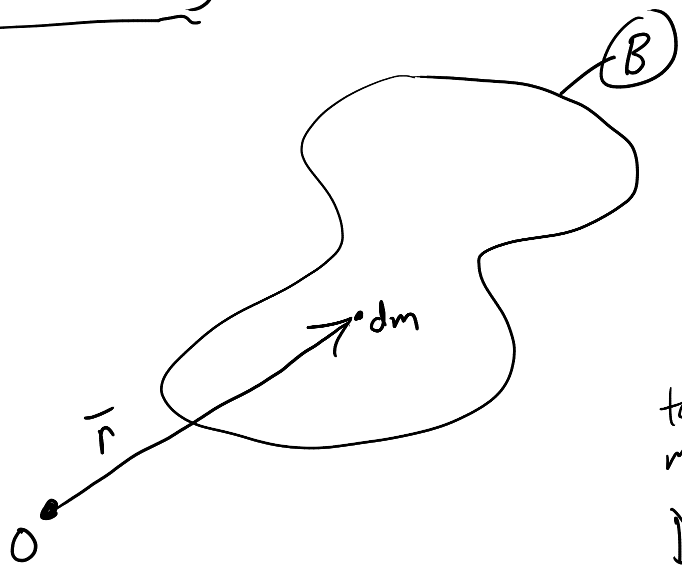
Mass & Mass Distribution

Particle

Point that has mass. Mass is the resistance to linear acceleration.

Rigid Body



Continous distribution of mass bounded by a volume, surface, or a curve.

$$m = \int dm \qquad m = \int \rho(\bar{r})\, dV$$

total mass

Distribution of mass resists angular acceleration.

Why important?

Equations of motion of single particle & rigid body:

Newton's second law

$$\sum \overline{F} = \frac{d\overline{p}}{dt}$$

$\overline{p} \triangleq$ linear momentoom

$\overline{p} = m\overline{v}$

m ↑ mass

Euler's second law

$$\sum \overline{M} = \frac{d\overline{H}}{dt}$$

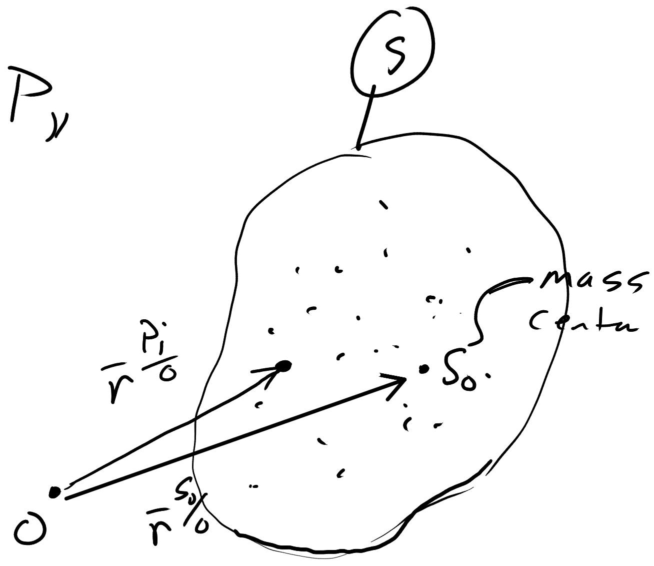
$\overline{H} \triangleq$ angular momentum

$\overline{H} = \underset{\smile}{I} \cdot \overline{\omega}$

↑ inertia (not scalar!)

Mass Center

For set S of ν particles $P_1, \dots, P_\nu$



$$m = \sum_{i=1}^{\nu} m_i$$ "zeroth moment of mass"

↑ total mass

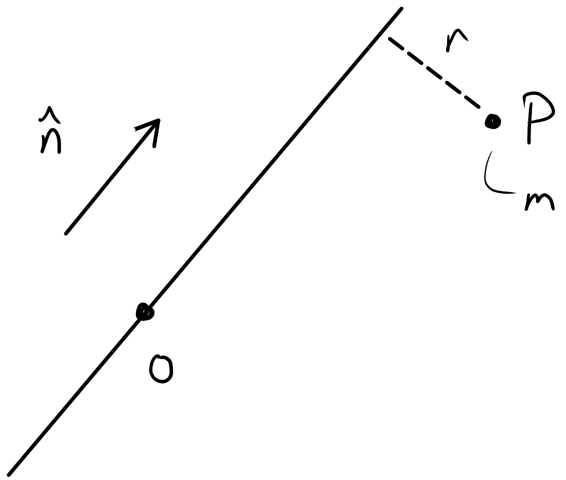
$$\sum_{i=1}^{\nu} m_i \bar{r}^{P_i/O}$$ "first moment of mass"

$$\sum_{i=1}^{\nu} m_i \bar{r}^{P_i/S_o} = 0$$

$$\sum_{i=1}^{\nu} m_i \left(\bar{r}^{P_i/O} - \bar{r}^{S_o/O}\right) = 0 \Rightarrow \sum m_i \bar{r}^{P_i/O} = \sum m_i \bar{r}^{S_o/O}$$

$$\bar{r}^{S_o/O} = \frac{\sum_{i=1}^{\nu} m_i \bar{r}^{P_i/O}}{\sum_{i=1}^{\nu} m_i}$$ equation for the mass center

Mass Distribution

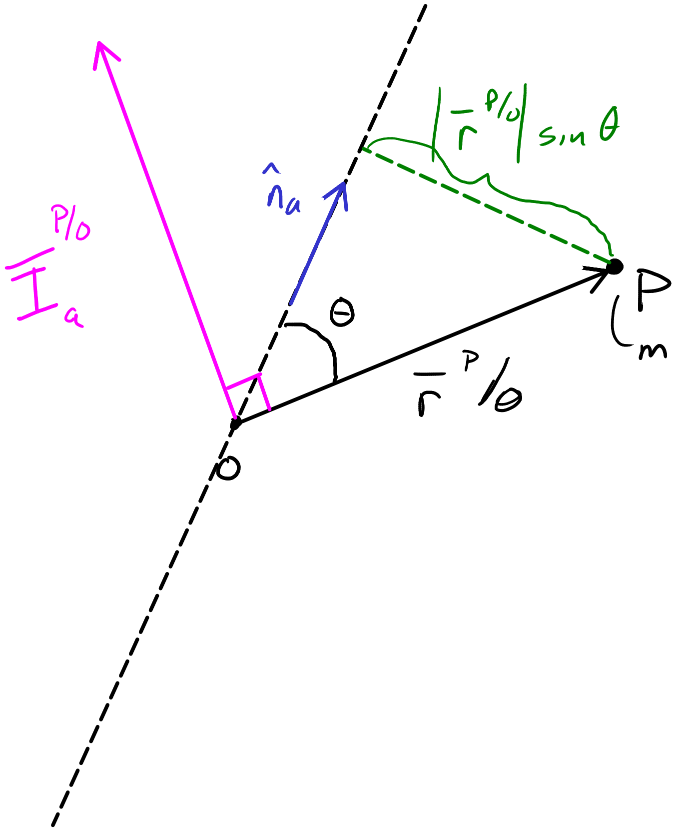


second moment of mass

mr^2

more generally, an <u>inertia vector</u> for a set of particles S with respect to point O about $\hat{n}_a$

$$\bar{I}_a^{S/O} \triangleq \sum_{i=1}^{\nu} m_i \, \bar{r}^{P_i/O} \times \left(\hat{n}_a \times \bar{r}^{P_i/O} \right)$$



$$|\hat{n}_a \times \bar{r}^{P/o}| = |\bar{r}^{P/o}| \sin\theta$$

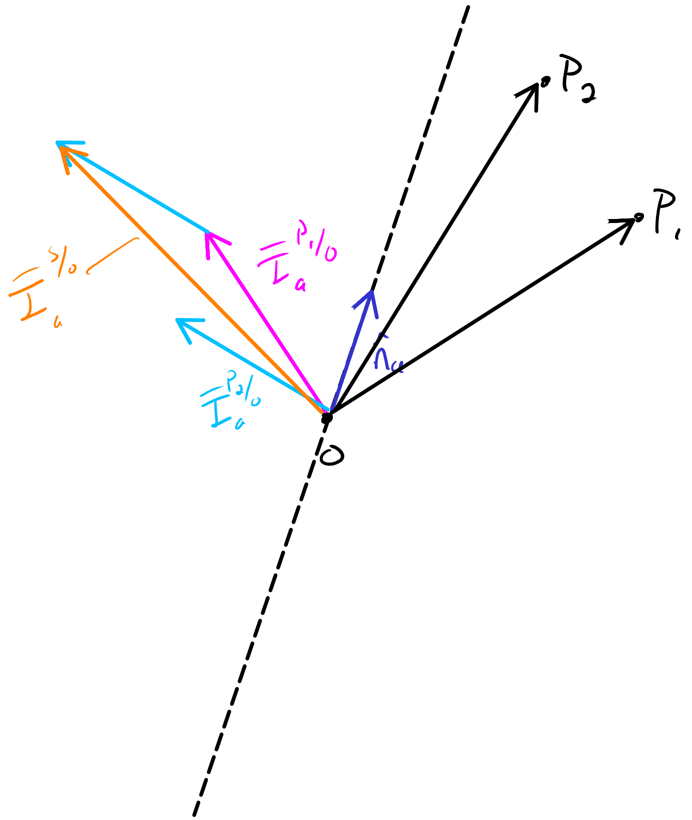
$$|\bar{I}_a^{P/o}| = m|\bar{r}^{P/o}|^2 \sin\theta$$

if $\theta = 0$

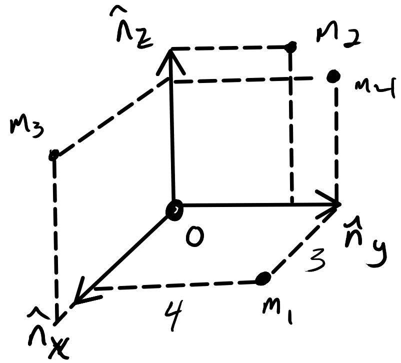
$$|\bar{I}_a^{P/o}| = 0$$

if $\theta = 90^\circ$

$$|\bar{I}_a^{P/o}| = m|\bar{r}^{P/o}|^2$$



Inertia vector contains the full information about the mass distribution of the set of particles



m	x	y	z
m_1	3	4	0
m_2	0	3	4
m_3	4	0	3
m_4	0	4	3

$$\bar{\bar{I}}_x^{S/O} = (16m_1 + 25m_2 + 9m_3 + 25m_4)\hat{n}_x - 12m_1\hat{n}_y - 12m_3\hat{n}_z$$

Inertia scalars

$$I_{ab}^{S/O} \triangleq \bar{\bar{I}}_a^{S/O} \cdot \hat{n}_b$$

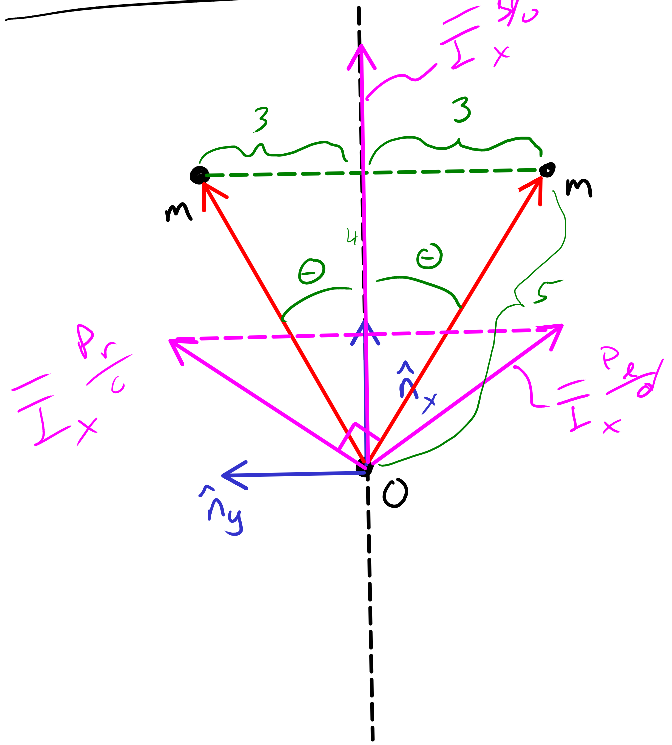
if $\hat{n}_b = \hat{n}_a$ => moment inertia scalar

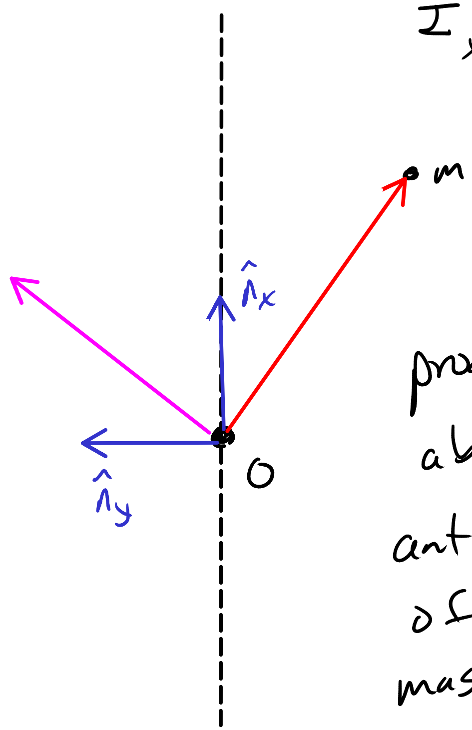
if $\hat{n}_b \neq \hat{n}_a$ => product inertia scalar

$$\bar{\bar{I}}_x^{S/O} \cdot \hat{n}_x = 16m_1 + 25m_2 + 9m_3 + 25m_4$$ → moment of inertia about $\hat{n}_x$ wrt O

$$\bar{I}_x^{S/O} \cdot \hat{n}_y = -12m_1$$ xy product of inertia wrt O

Moment of inertia and product of inertia





$$\bar{\bar{I}}_x^{P_r/O} = 9m\,\hat{n}_x + 12m\,\hat{n}_y$$

$$\underbrace{\bar{\bar{I}}_x^{P_r/O} \cdot \hat{n}_y = 12m}_{\text{product of inertia}}$$

product tells about the antisymmetry of the mass distribution!