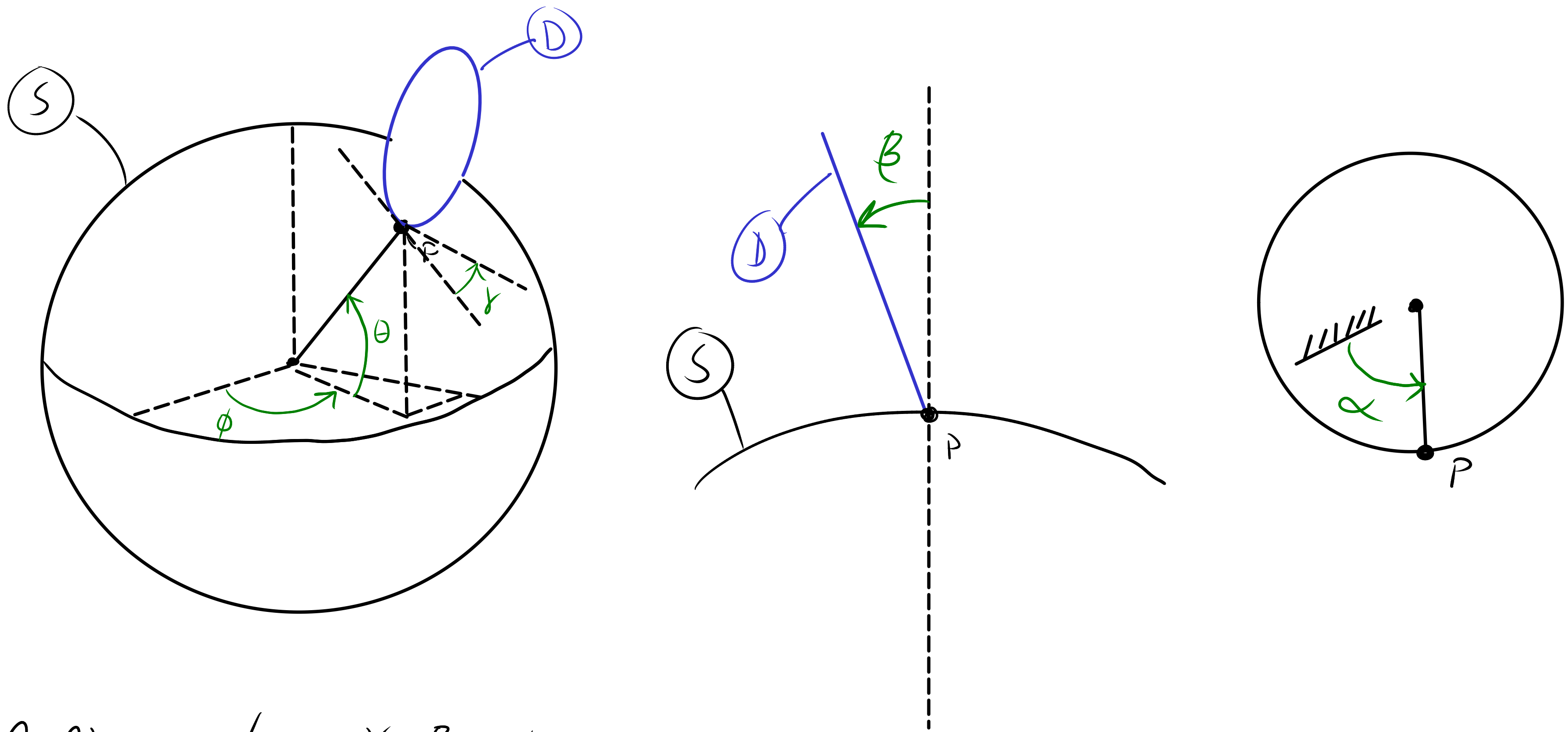
Degrees of Freedom

Degrees of freedom indicate the number of independent ways that a system is free to move in a reference frame. For holonomic system there are n degrees of freedom, one for each generalized coordinate. Nonholomic constraints further reduce the degrees of freedom. For a system with n generalized coordinates in RF A with m nonholomic constraints the p degrees of freedom are:

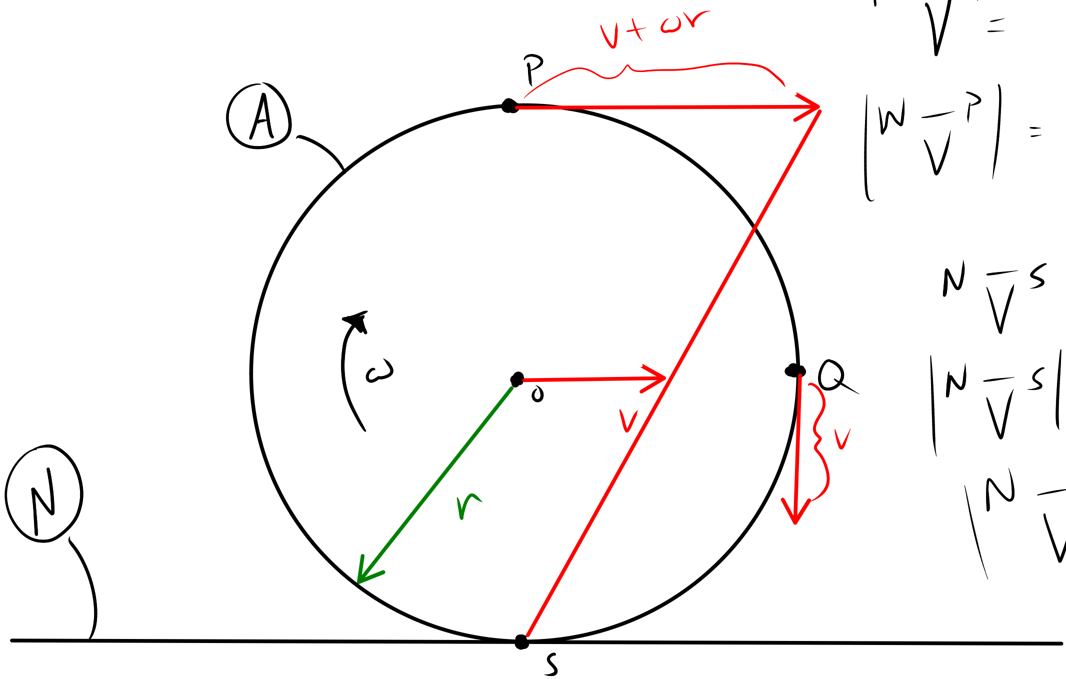
$$\underbrace{p}_{\text{DoF}} \triangleq \overbrace{n}^{\#\text{ GCs}} - \underbrace{m}_{\#\text{ nonholonomic constraints}}$$

Example: Disc rolling on sphere



5 G.C's : $\phi, \theta, \gamma, \beta, \alpha$ $\qquad n = 5$

Rolling without slip



$$^N\bar{V}^P = {}^N\bar{V}^O + {}^N\bar{\omega}^A \times \bar{r}^{P/O}$$

$$|{}^N\bar{V}^P| = v + \omega r$$

$$^N\bar{V}^S = {}^N\bar{V}^O + {}^N\bar{\omega}^A \times \bar{r}^{S/O}$$

$$|{}^N\bar{V}^S| = v - \omega r = 0 \Rightarrow v = \omega r$$

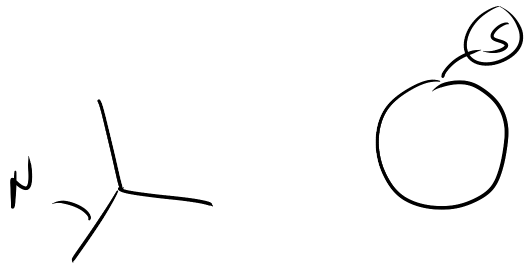
$$|{}^N\bar{V}^P| = 2v = 2\omega r$$

If the wheel rolls without slip on the disk there are $m=2$ nonholomic constraints.

If P is the point fixed in D that is at the contact point between the disc and the sphere then ${}^{S}\bar{V}^{P}=0$. There are 2 components of velocity in the tangent plane at the contact point.

So $p = 5-2 = 3$ degrees of freedom.

If the sphere is moving freely in N:



there are 6 GCs to locate and orient the sphere in N

$p = 11-2 = 9$ DoF

Independent and depedent generalized speeds

With generalized speeds $\bar{u}$ defined the nonholomic constraints can be written as

$$\bar{f}_n(\bar{u},\bar{q},t) = 0 \qquad \bar{f}_n \in \mathbb{R}^m \qquad \bar{u},\bar{q} \in \mathbb{R}^n$$

The nonholomic constraints imply that some of the generalized speeds are dependent on the others. m generalized speeds can be selected as dependendent generalized speeds $\bar{u}_r$, leaving p independent generalized speeds $\bar{u}_s$.

$$\bar{f}_n(\bar{u}_r,\bar{u}_s,\bar{q},t) = 0 \qquad \bar{u}_r \in \mathbb{R}^m \qquad \bar{u}_s \in \mathbb{R}^{p=n-m}$$

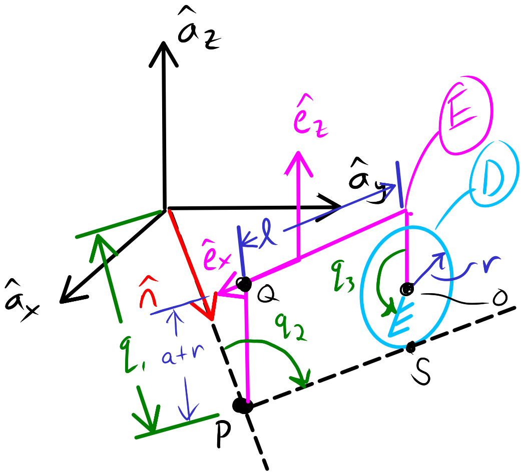
$$\bar{f}_n = A_r\bar{u}_r - A_s\bar{u}_s - \bar{b}_{rs} = 0$$

$$\bar{u}_r = A_r^{-1}\left(A_s \bar{u}_s + \bar{b}_{rs}\right) = A_n \bar{u}_s + \bar{b}_n$$

$$A_n = A_r^{-1} A_s$$

$$\bar{b}_n = A_r^{-1} \bar{b}_{rs}$$

Example



Simple trailer E has wheel D that rolls without slip on the plane normal to $\hat{a}_z$. Disc D is perpendicular to the ground plane. The hitch P is pulled in the $\hat{n}$ direction.

$$\hat{n} = c_2 \hat{e}_x + s_2 \hat{e}_y$$

Generalized coordinates: q_1, q_2, q_3

$$\cos q_i = c_i$$

$$\sin q_i = s_i$$

$$^A\bar{V}^P = \dot{q}_1 \hat{n} \qquad {}^A\bar{\omega}^E = -\dot{q}_2 \hat{a}_z$$

$$^A\bar{V}^Q = \dot{q}_1 \hat{n} \qquad {}^A\bar{\omega}^D = {}^A\bar{\omega}^E + {}^E\bar{\omega}^D$$

$$^E\bar{\omega}^D = \dot{q}_3 \hat{e}_y$$

$$^A\bar{V}^O = {}^A\bar{V}^Q + {}^A\bar{\omega}^E \times \bar{r}^{O/Q}$$

$$= \dot{q}_1 \hat{n} - \dot{q}_2 \hat{e}_z \times (-\ell \hat{e}_x - a\hat{e}_z)$$

$$^A\bar{V}^O = \dot{q}_1 c_2 \hat{e}_x + (\dot{q}_1 s_2 + \ell \dot{q}_2) \hat{e}_y$$

$${}^{A}\bar{V}^{S} = {}^{A}\bar{V}^{O} + {}^{N}\bar{\omega}^{E} \times \bar{r}^{S/O}$$

$$= {}^{A}\bar{V}^{O} + (\dot{q}_3 \hat{e}_y - \dot{q}_2 \hat{e}_z) \times (-r\hat{e}_z)$$

$${}^{A}\bar{V}^{S} = (\dot{q}_1 c_2 - r\dot{q}_3)\hat{e}_x + (\dot{q}_1 s_2 + l\dot{q}_2)\hat{e}_y$$

rolling without slip

$${}^{A}\bar{V}^{S} = 0$$

$\dot{q}_1 c_2 - r\dot{q}_3 = 0$ and $\dot{q}_1 s_2 + l\dot{q}_2 = 0$

$m = 2$ nonholonomic constraints

$p = 3 - 2 = 1$ DoF !

define generalized speeds:

$$u_1 = \dot{q}_1 \qquad u_2 = {}^A\bar{V}^S \cdot \hat{e}_x = \dot{q}_1 c_2 - r\dot{q}_3 \qquad u_3 = {}^A\bar{V}^S \cdot \hat{e}_y = \dot{q}_1 s_2 + l\dot{q}_2$$

$$\bar{u} = Y_k \dot{\bar{q}} + \bar{Z}_k \qquad \bar{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \qquad \bar{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$Y_k = \begin{bmatrix} 1 & 0 & 0 \\ c_2 & 0 & -r \\ s_2 & l & 0 \end{bmatrix} \qquad \bar{Z}_k = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

solve for $\dot{\bar{q}}$

$$\dot{q}_1 = u_1 \qquad \dot{q}_2 = -u_1 s_2 / l + u_3 \qquad \dot{q}_3 = u_1 c_2 / r - u_2$$

Now rewrite nonholonomic constraints in terms of the u's:

$$A_r = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad A_s = \begin{bmatrix} 0 \end{bmatrix} \qquad \bar{b}_{rs} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\bar{f}_n = \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} u_2 \\ u_3 \end{bmatrix} - \begin{bmatrix} 0 \end{bmatrix}\begin{bmatrix} u_1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\dot{q}_1 = u_1 \qquad \dot{q}_2 = -u_1 s_2/l \qquad \dot{q}_3 = u_1 c_2/r$

rewrite velocities

$^A\bar{V}^P = {}^A\bar{V}^Q = u_1 \hat{n} \qquad {}^A\bar{V}^O = u_1 c_2 \hat{e}_x \qquad {}^A\bar{V}^S = 0$

$^A\bar{\omega}^E = -u_1 s_2/l \, \hat{e}_z \qquad {}^A\bar{\omega}^P = -u_1/s_2/l \, \hat{e}_z + u_1 c_2/r \, \hat{e}_y$

$\bar{U}_r = \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} \qquad \bar{U}_s = \begin{bmatrix} u_1 \end{bmatrix}$

Velocities only in terms of the generalized speeds.