Generalized Speeds & Kinematical Differential Equations

Newton's Second Law for particle

$\sum \bar{F} = m\bar{a} \xrightarrow[\text{scalar}]{} F = ma = m\ddot{x} \Rightarrow$ second order ordinary differential equation in x

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$$\left.\begin{array}{ll} ① & F = m\dot{v} \\ ② & v = \dot{x} \end{array}\right\} \text{two first order " " " in } x, v$$

. We will introduce n generalized speeds $u_1, \dots, u_n$.

$\bar{u} \in \mathbb{R}^n$ such that:

$$\bar{u} \triangleq Y_k \dot{\bar{q}} + \bar{Z}_k$$

Generalized speeds must be linear in $\dot{q}$'s and Y_k must be invertible, such that we can solve for the $\dot{q}$'s:

$$\dot{q} = Y_k^{-1}(\bar{u} - \bar{z}_k)$$

These called the "Kinematical differential equations"

Choosing generalized speeds

For the Chapylgin sleigh set $x = q_1$, $y = q_2$, $\theta = q_3$

$$^N\bar{V}^P = (\dot{q}_1 c_3 + \dot{q}_2 s_3)\hat{a}_x + (-\dot{q}_1 s_3 + \dot{q}_2 c_3)\hat{a}_y$$

$\sin q_i = s_i$ $\qquad$ $^N\bar{\omega}^A = \dot{q}_3 \hat{a}_z$

$\cos q_i = c_i$

A choice

$$\bar{u} = \dot{q}$$

$$^N\bar{V}^P = (u_1 c_3 + u_2 s_3)\hat{a}_x + (-u_1 s_3 + u_2 c_3)\hat{a}_y$$

$$^N\bar{\omega}^A = u_3 \hat{a}_z$$

$$Y_k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \qquad \bar{Z}_k = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{u} = I\dot{\bar{q}}$$

Another choice

$$u_1 = {}^N\bar{V}^P \cdot \hat{a}_x \;, \quad u_2 = {}^N\bar{V}^P \cdot \hat{a}_y \;, \quad u_3 = {}^N\bar{\omega}^A \cdot \hat{a}_z$$

$$\begin{cases} u_1 = \dot{q}_1 c_3 + \dot{q}_2 s_3 \\ u_2 = -\dot{q}_1 s_3 + \dot{q}_2 c_3 \\ u_3 = \dot{q}_3 \end{cases} \qquad \bar{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad Y_k = \begin{bmatrix} c_3 & s_3 & 0 \\ -s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \bar{Z}_k = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solve for $\dot{\bar{q}}$: (Y_k is invertible)

$$\dot{q}_1 = u_1 c_3 - u_2 s_3$$

$$\dot{q}_2 = u_1 s_3 + u_2 c_3$$

$$\dot{q}_3 = u_3$$

In practice, choosing the u's to be components of problem important velocities and angular velocities can simplify acceleration and angular accerleration expressions. But you can always select $\bar{u} = \dot{\bar{q}}$.