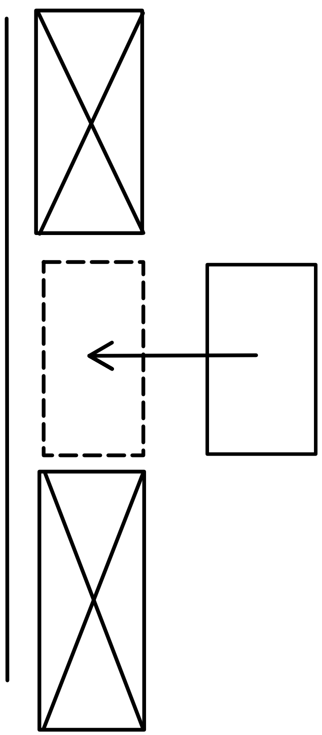
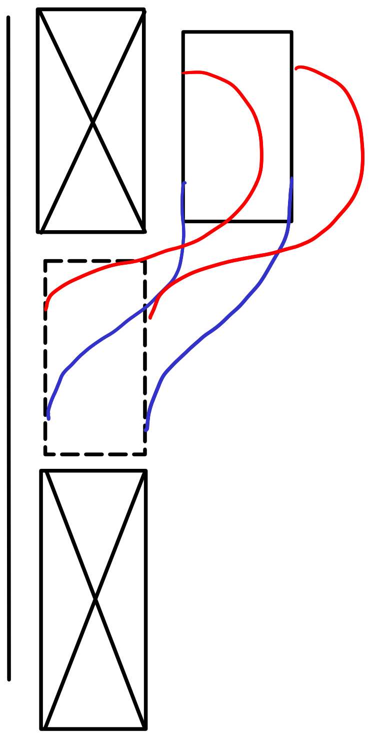
Nonholomic Constraints (Motion Constraints)

Example: parallel parking





Holomic constraints (configuration) $\rightarrow$ constraints:
location of points
orientation of RFs

$$\bar{f}_h(q_1, \ldots, q_N, t) = 0$$

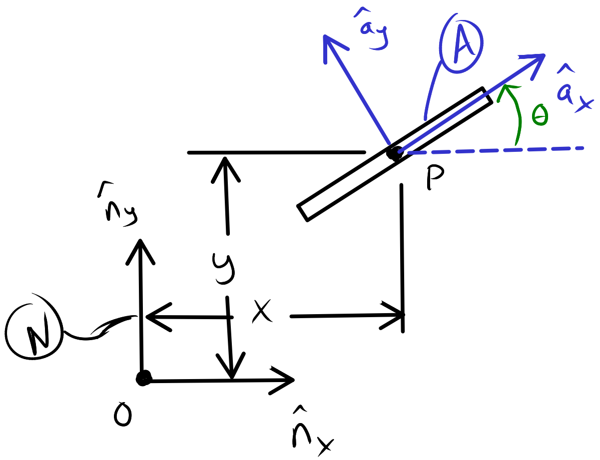
Nonholomic constraints (motion) $\rightarrow$ constraints on
velocity of points
angular velocity of RFs

$$\bar{f}_n(\dot{\bar{q}}, \bar{q}, t) = 0$$

$\bar{q}, \dot{\bar{q}} \in \mathbb{R}^n$ $\quad$ $\bar{f}_n \in \mathbb{R}^m$ $\quad$ m nonholomic constraints

Example: Chaplygin Sleigh

simplest model ice skate, ski on snow, runner of sleigh on snow



3 generalized coordinates: x, y, θ

$$\bar{r}^{P/O} = x\,\hat{n}_x + y\,\hat{n}_y$$

$$^N\bar{V}^P = \dot{x}\,\hat{n}_x + \dot{y}\,\hat{n}_y$$

$$^N\bar{\omega}^A = \dot{\theta}\,\hat{n}_z = \dot{\theta}\,\hat{a}_z$$

$$^N\bar{V}^P \cdot \hat{a}_y = 0 \Rightarrow \left[(\dot{x}\cos\theta + \dot{y}\sin\theta)\,\hat{a}_x + \underbrace{(-\dot{x}\sin\theta + \dot{y}\cos\theta)}\,\hat{a}_y\right] \cdot \hat{a}_y = 0$$

$$\hat{n}_x = \cos\theta\,\hat{a}_x - \sin\theta\,\hat{a}_y$$
$$\hat{n}_y = \sin\theta\,\hat{a}_x + \cos\theta\,\hat{a}_y$$

$$\boxed{-\dot{x}\sin\theta + \dot{y}\cos\theta = 0}$$

one $(m=1)$ nonholomic constraint

$$\dot{x}\tan\theta - \dot{y} = 0$$

Imagine we have a configuration constraint $f_h = 0$ that looks like:

$$\cos q_1 - l \sin q_2 = 0 = f_h(q_1, q_2)$$

Differentiate wrt time:

$$-\sin q_1 \cdot \dot{q}_1 - l \cos q_2 \cdot \dot{q}_2 = 0$$

looks like nonholomic constraint !

$$f_n(\dot{q}_1, \dot{q}_2, q_1, q_2) = 0$$

But we know $\int f_n \, dt = f_h$

f_h is a holomic constraint in disguise.

To see if $\bar{f}_n$ is integrable you can do this check.

$$f_n = \frac{df_h}{dt}$$ then can write:

$$\frac{df_h}{dt} = \frac{\partial f_h}{\partial q_1}\frac{dq_1}{dt} + \dots + \frac{\partial f_h}{\partial q_N}\frac{dq_N}{dt} + \frac{\partial f_h}{\partial t}$$

For the Chaplygin Sleigh:

$$f_n = \frac{df_h}{dt} = \frac{\partial f_h}{\partial x}\dot{x} + \frac{\partial f_h}{\partial y}\cdot\dot{y} + \frac{\partial f_h}{\partial \theta}\dot{\theta}$$

$$f_n = \dot{x}\tan\theta - \dot{y} = 0$$

$$\frac{\partial f_h}{\partial x} = \tan\theta \qquad \frac{\partial f_h}{\partial y} = -1 \qquad \frac{\partial f_h}{\partial \theta} = 0$$

According to Schwarz/Clairant/Young's therom tells us that f_n is not integrable if the mixed partial do not commute:

$$\frac{\partial^2 f_h}{\partial x \partial y} \neq \frac{\partial^2 f_h}{\partial y \partial x}, \quad \frac{\partial^2 f_h}{\partial \theta \partial y} \neq \frac{\partial^2 f_h}{\partial y \partial \theta}, \quad \frac{\partial^2 f_h}{\partial x \partial \theta} \neq \frac{\partial^2 f_h}{\partial \theta \partial x}$$

$$\begin{array}{cccccc} \| & \| & \| & \| & \| & \| \\ 0 & 0 & 0 & 0 & 0 & \sec^2\theta \end{array}$$

integrable   integrable   not integrable

$\dot{x}\tan\theta - \dot{y} = 0$ is an essential nonholonomic constraint!