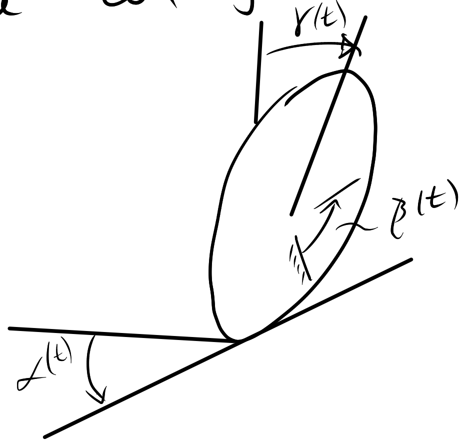
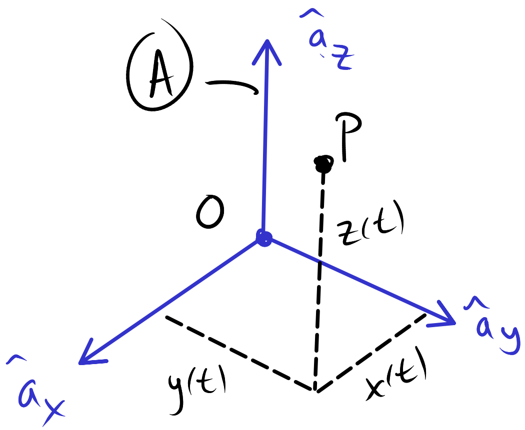
Holonomic Constraints

We've been "configuration variable" to describe the configuration of multibody systems.



$\alpha(t), \gamma(t), \beta(t)$ are configuration variables

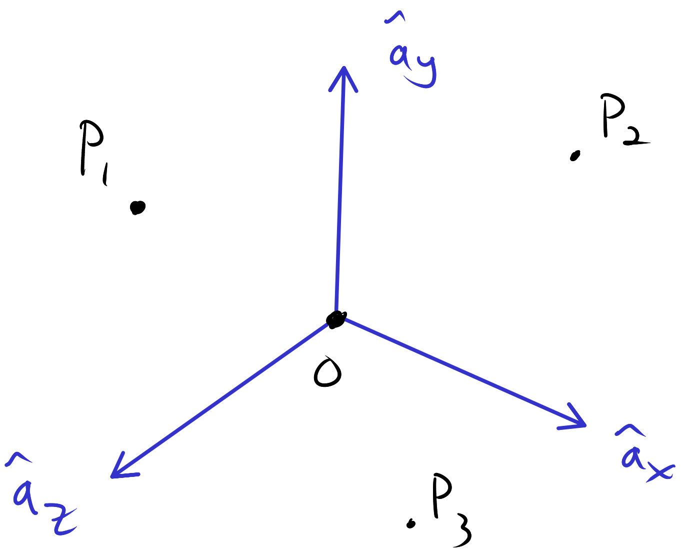
In general each point in 3D space can be described by three configuration variables: 3 Cartesian coordinates.



We can constrain the configuration variables by specifying some of these coordinates.

For example if $z(t) = 0$ then the point P would be constrained to motion in the x-y plane. And if $x(t) = y(t) = z(t) = 0$ the point P is fully constrained (fixed in A). If we have a collection of ν points, then it requires 3ν constraints to fix all points in Euclidean space.

Take for example 3 points $\underbrace{P_1, P_2, P_3}_{\text{not fixed in } A}$ and a point zero O which in A.



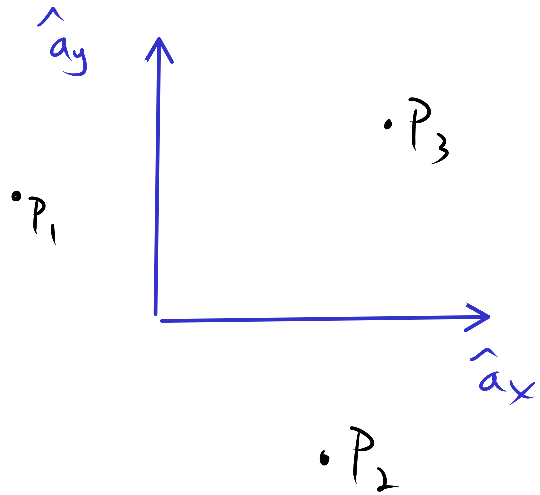
To constrain all points we need $3 \cdot 3 = 9$ constraints.

First, constrain all points to move in a plane (x-y plane):

① $\bar{r}^{P_1/0} \cdot \hat{a}_z = 0$

② $\bar{r}^{P_2/0} \cdot \hat{a}_z = 0$

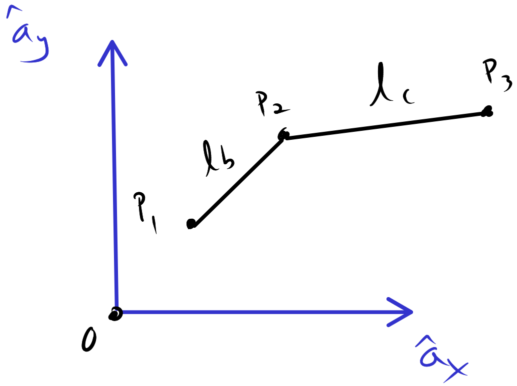
③ $\bar{r}^{P_3/0} \cdot \hat{a}_z = 0$



Now constrain points to have a specified distance among some of them.

④ $\left|\bar{r}^{P_2/P_1}\right| = l_b$

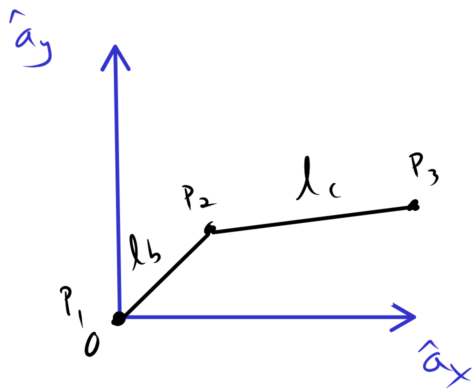
⑤ $\left|\bar{r}^{P_3/P_2}\right| = l_c$



Now constrain P_1 to be fixed in A.

⑥ $$\bar{r}^{P_1/O} \cdot \hat{a}_x = 0$$

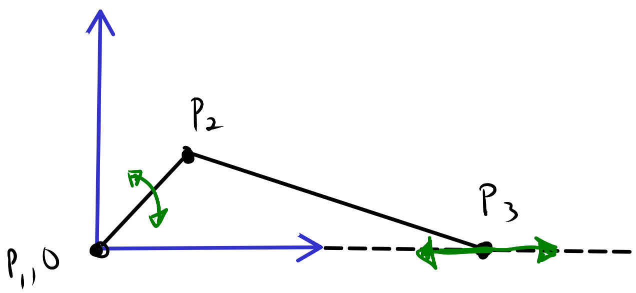
⑦ $$\bar{r}^{P_1/O} \cdot \hat{a}_y = 0$$



Now constrain point P_3 to have no motion in y direction.

(8) $\bar{r}^{P_3/O} \cdot \hat{a}_y = 0$

planar
crank-slider
mechanism



Started with $3 \cdot 3 = 9$ cartesian coordinates and we've introduced $3+2+2+1=8$ scalar constraint equations. We are left with a single independent coordinate

$$n = 3\nu - M$$

$$n = 3 \cdot 3 - 8 = \underline{1}$$

In general, a set of ν points can be constrained by M constraints leaving $n = 3\nu - M$ independent configuration variables. The constraints take this general form:

These are called **holonomic constraints**:

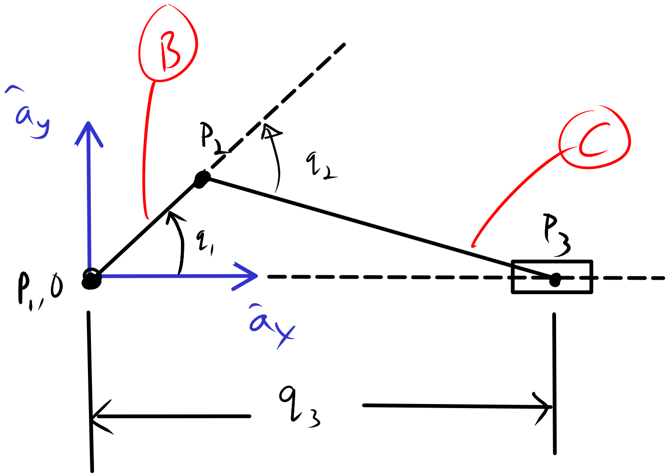
$$\bar{f}_h(x_1, y_1, z_1, \dots, x_\nu, y_\nu, z_\nu, t) = 0 \quad \text{where } \bar{f}_h \in \mathbb{R}^M$$

Explicit: rheonomic holonomic con.

t (time) can be explicit in the equations. Implicit: scleronomic " "

But there is a better way! Most constraints can be implicitly managed if we describe our system initially with the configuration we desire.

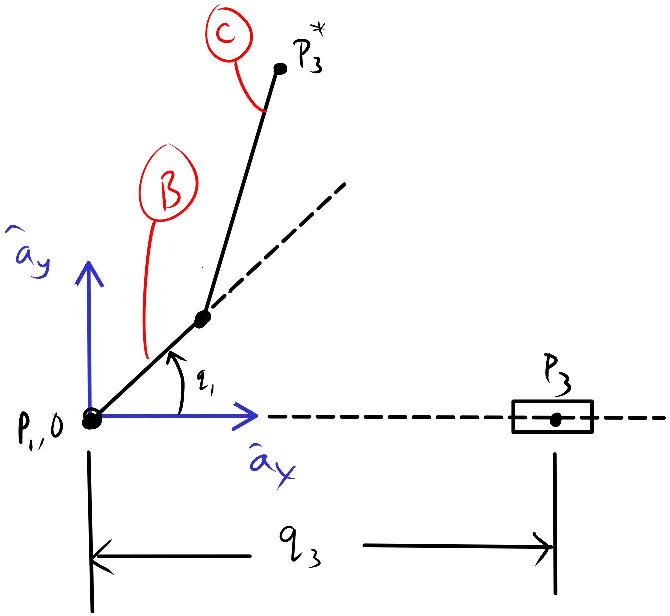
Called <u>closed</u> <u>kinematic</u> <u>loop</u> because:



$$\bar{r}^{P_3/P_1} = \bar{r}^{P_2/P_1} + \bar{r}^{P_3/P_2}$$

This equation describes holonomic constraints of this configuration.

N configuration variables, $N=3$



$$\bar{r}^{P_3/P_1} = q_3\,\hat{a}_x$$

$$\bar{r}^{P_2/P_1} + \bar{r}^{P_3^*/P_2} = l_b\,\hat{b}_x + l_c\,\hat{c}_x$$

$$\boxed{q_3\,\hat{a}_x - l_b\,\hat{b}_x - l_c\,\hat{c}_x = 0}$$

vector form of holonomic constraint

To arrive at 2 holonomic scalar constraints:

$$\left(q_3 \hat{a}_x - l_b \hat{b}_x - l_c \hat{c}_x\right) \cdot \hat{a}_x = 0$$

P_3 & P_3^* must have the same x value

$$\left(q_3 \hat{a}_x - l_b \hat{b}_x - l_c \hat{c}_x\right) \cdot \hat{a}_y = 0$$

P_3^* must lie on x axis

an alternative is to express in B.

$$q_3 \cos q_1 \hat{b}_x - q_3 \sin q_1 \hat{b}_y - l_b \hat{b}_x - l_c \cos q_2 \hat{b}_x - l_c \sin q_2 \hat{b}_y = 0$$

$$\boxed{\begin{array}{ll} \hat{b}_x: & q_3 \cos q_1 - l_b - l_c \cos q_2 = 0 \\ \hat{b}_y: & -q_3 \sin q_1 - l_c \sin q_2 = 0 \end{array}}$$

2 scalar holonomic constraints

We had $N=3$ coordinates (q_1, q_2, q_3) and we have $M=2$ constraints. That $n = N - M = 3 - 2 = 1$ independent coordinates.

If we choose q_1 to be independent then q_2 & q_3 would be dependent on q_1.

$$\bar{f}_h(q_1, q_2, q_3) = 0 \quad \text{where } \bar{f}_h \in \mathbb{R}^2$$

$$\bar{f}_h = \begin{bmatrix} q_3 c_1 - l_b - l c_2 \\ \\ -q_3 s_1 - l s_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \\ 0 \end{bmatrix}$$

Note that we only need 2 constraints instead of 8 as before. This is because we choose coordinates q_1, q_2, q_3 and had constraints implicitly defined in our system description. Same conclusion: 1 independent coordinate fully describes the configuration. We can choose q_1, q_2, <u>or</u> q_3.

There exists coordinates $q_1(t), \ldots, q_n(t)$ that are functions of the 3ν Cartesian coordinates that uniquely describe the configuration of the system. These n coordinates are independent coordinates that minimize the number of constraints needed to describe the system. These $\underline{n}$ coordinates are <u>generalized</u> <u>coordinates</u>.

For the crank slider, if we select q_1 as the independent coordinate, then q_1 is our generalized coordinate. And q_2 and q_3 are not G.C.s!