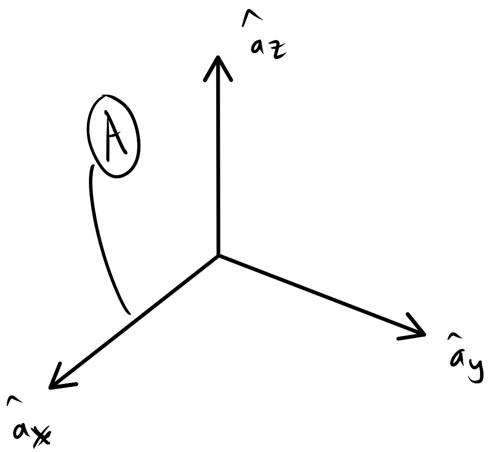
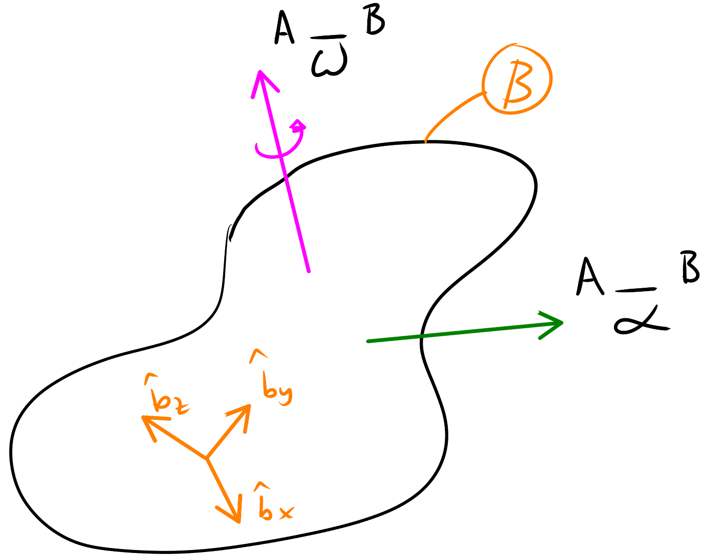
Angular Kinematics





Angular Velocity

Definition of angular velocity of a reference frame B when the change in orientation is observed from reference frame A.

$$^{A}\underset{\omega}{-}^{B} \triangleq \left(\frac{^{A}d\hat{b}_y}{dt}\cdot\hat{b}_z\right)\hat{b}_x + \left(\frac{^{A}d\hat{b}_z}{dt}\cdot\hat{b}_x\right)\hat{b}_y + \left(\frac{^{A}d\hat{b}_x}{dt}\cdot\hat{b}_y\right)\hat{b}_z$$

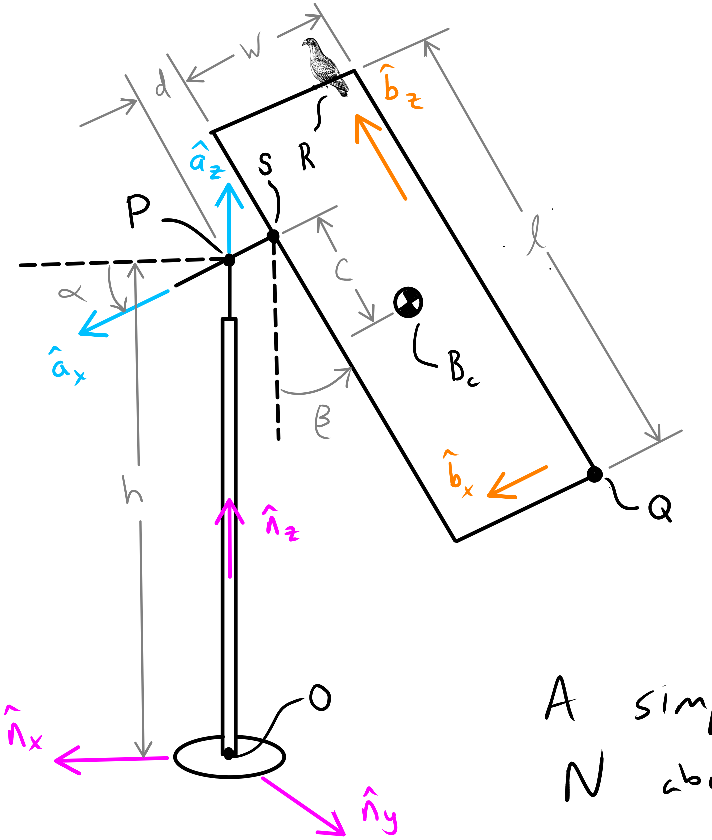
Apply this definition by expressing $\hat{b}_x, \hat{b}_y, \hat{b}_z$ in A and taking time derivative wrt A.

$$\hat{b}_x = a_1\hat{a}_x + a_2\hat{a}_y + a_3\hat{a}_z$$

$$\frac{^{A}d\hat{b}_x}{dt} = \dot{a}_1\hat{a}_x + \dot{a}_2\hat{a}_y + \dot{a}_3\hat{a}_z$$

More useful theorems that arrive from the definition.

Motivating example: kinectic sculpture

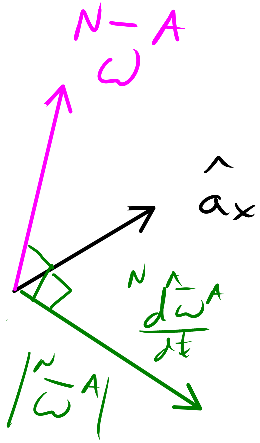


Theorem 1 Ang. Vel.

$$\frac{{}^N d\hat{a}_x}{dt} = {}^N\bar{\omega}^A \times \hat{a}_x$$

if

$\hat{a}_x$ is fixed in A



A simply orients wrt N about $\hat{a}_z = \hat{n}_z$ through α.

$${}^N\bar{\omega}^A = \dot{\alpha}\hat{n}_z = \dot{\alpha}\hat{a}_z$$

$$\frac{{}^N d\hat{a}_x}{dt} = \dot{\alpha}\hat{a}_z \times \hat{a}_x = \dot{\alpha}\hat{a}_y$$

Theorem 2 Ang. Vel.

Vector $\bar{r}^{S/P} = -d\,\hat{a}_x$ is fixed in A.

d is constant

$$^N\frac{d\bar{r}^{S/P}}{dt} = {}^N\bar{\omega}^A \times \bar{r}^{S/P}$$

$$= \dot{\alpha}\, a_z \times -d\,\hat{a}_x = -d\dot{\alpha}\,\hat{a}_y$$

Theorem 3 Ang. Vel.

Suppose d is not constant, i.e. $d(t)$

$$^N\frac{d\bar{r}^{S/P}}{dt} = {}^A\frac{d\bar{r}^{S/P}}{dt} + {}^N\bar{\omega}^A \times \bar{r}^{S/P}$$

$$\bar{r}^{S/P} = -d\,\hat{a}_x$$

$$\frac{{}^{A}d\bar{r}^{S/P}}{dt} = -\dot{d}\,\hat{a}_x$$

$$\frac{{}^{N}d\bar{r}^{S/P}}{dt} = -\dot{d}\,\hat{a}_x - d\dot{\alpha}\,\hat{a}_y$$

Theorem 4 Ang. Vel.

$${}^{N}\bar{\omega}^{B} = {}^{N}\bar{\omega}^{A} + {}^{A}\bar{\omega}^{B}$$

$${}^{N}\bar{\omega}^{A} = \dot{\alpha}\,\hat{a}_z \qquad {}^{A}\bar{\omega}^{B} = \dot{\beta}\,\hat{a}_x$$

so

$${}^{N}\bar{\omega}^{B} = \dot{\alpha}\,\hat{a}_z + \dot{\beta}\,\hat{a}_x$$

Angular Acceleration

Definition angular acceleration of A when observed from N.

$$^N\bar{\alpha}^A \triangleq \frac{^N d\, ^N\bar{\omega}^A}{dt}$$

$$^N\bar{\omega}^A = \dot{\alpha}\,\hat{n}_z = \dot{\alpha}\,\hat{a}_z$$

$$\frac{^N d}{dt}(\dot{\alpha}\,\hat{n}_z) = \ddot{\alpha}\,\hat{n}_z$$

Theorem 1 Ang. Acc.

$$\frac{^N d\, ^N\bar{\omega}^A}{dt} = \frac{^A d\, ^N\bar{\omega}^A}{dt}$$

Same no matter what (A or N) you observe the change from.

$$\frac{^A d\, ^N\bar{\omega}^A}{dt} = \frac{^A d}{dt}(\dot{\alpha}\,\hat{a}_z) = \ddot{\alpha}\,\hat{a}_z = \ddot{\alpha}\,\hat{n}_z$$

Theorem 2 Ang. Acc.

$$ {}^N\bar{\alpha}^B \neq {}^N\bar{\alpha}^A + {}^A\bar{\alpha}^B $$

You cannot sum angular accelerations for successive orientations !

* $${}^N\bar{\alpha}^A = \ddot{\alpha}\,\hat{a}_z = \ddot{\alpha}\sin\beta\,\hat{b}_y + \ddot{\alpha}\cos\beta\,\hat{b}_z$$
* $${}^A\bar{\alpha}^B = \ddot{\beta}\,\hat{b}_x = \ddot{\beta}\,\hat{a}_x$$

$$ {}^N\bar{\omega}^B = \dot{\alpha}\,\hat{a}_z + \dot{\beta}\,\hat{a}_x $$

$$ {}^N\bar{\omega}^B = \dot{\alpha}\sin\beta\,\hat{b}_y + \dot{\alpha}\cos\beta\,\hat{b}_z + \dot{\beta}\,\hat{b}_x $$

$$ \frac{{}^N d\,{}^N\bar{\omega}^B}{dt} = \frac{{}^B d\,{}^N\bar{\omega}^B}{dt} $$

$$ \frac{{}^N d\,{}^N\bar{\omega}^B}{dt} = \ddot{\beta}\,\hat{b}_x + \left(\ddot{\alpha}\sin\beta + \dot{\alpha}\cos\beta\,\dot{\beta}\right)\hat{b}_y + \left(\ddot{\alpha}\cos\beta - \dot{\alpha}\sin\beta\,\dot{\beta}\right)\hat{b}_z = {}^N\bar{\alpha}^B $$